Quaternion group based quantization of SL(4,R)

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Abstract

Dr R. A. Wilson proposes the finite–group quantisation of the real form $SL(4,\mathbb{R})$ electroweak sector should be quantised by a specific finite subgroup of $SO(4)$, built from two quaternion groups Q_8, together with distinct left– and right–handed triplet extensions. His construction yields a group of order 288, with a normal $Q_8 \times Q_8$ subgroup and quotient $S_3 \times \mathbb{Z}_3$, and is intended to encode parity violation, flavour structure, and the observed asymmetry between $SU(2)_L$ and $SU(2)_R$.

The continuous real–form chain

$$SL(4,\mathbb{R}) \supset SO(3,3) \supset SO(4) \simeq (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2,$$

arises in twistor–based unification. Using the quaternionic model of $SO(4)$, we construct explicit embeddings of $Q_8^{(L)}$ and $Q_8^{(R)}$ into $SU(2)_L$ and $SU(2)_R$, adjoin Wilson's broken and unbroken triplet symmetries, and obtain a canonical finite subgroup

$$H_{288} \subset SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4,\mathbb{R}),$$

realising his proposed quantisation at the electroweak stage. We also show how the non–commuting triplet structure of H_{288} aligns with the continuous strong–sector chain $SL(3,\mathbb{R}) \to SU(3)_{\text{eff}}$, in which colour is emergent rather than fundamental. Finally, we indicate how this finite subgroup fits into the broader twistor symmetry $SL(4,\mathbb{C})$, providing a discrete analogue of the $SL(4,\mathbb{R})$ branch complementary to the compact $SU(4)$ and conformal $SU(2,2)$ real forms.

1 SO(4) as $(SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$ and quaternion actions

We recall the standard identification

$$SO(4) \;\simeq\; \frac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}, \tag{1}$$

where $SU(2)_L$ and $SU(2)_R$ act on the quaternion space $\mathbb{H} \simeq \mathbb{R}^4$ by

$$(q_L, q_R) : \; x \;\mapsto\; q_L \, x \, q_R^{-1}, \qquad x \in \mathbb{H}, \tag{2}$$

with $q_L, q_R \in SU(2) \simeq \{q \in \mathbb{H} : |q| = 1\}$.

Writing $\mathbb{H}$ in the basis $\{1, i, j, k\}$, we identify

$$\mathbb{H} \;\simeq\; \mathbb{R}^4, \qquad x = x_0 + x_1 i + x_2 j + x_3 k \;\leftrightarrow\; (x_0, x_1, x_2, x_3)^T. \tag{3}$$

Left and right multiplication by unit quaternions then define real 4×4 matrices, giving an explicit embedding

$$SU(2)_L \times SU(2)_R \;\hookrightarrow\; SO(4) \;\hookrightarrow\; SL(4, \mathbb{R}). \tag{4}$$

For example, left multiplication by i acts as

$$L_i(x) = ix, \tag{5}$$

which in the $(1, i, j, k)$ basis is represented by

$$L_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{6}$$

and similarly for L_j, L_k and the right actions R_i, R_j, R_k.

2 Two copies of Q_8 inside $SU(2)_L \times SU(2)_R$

The quaternion group

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset SU(2) \tag{7}$$

has the usual relations

$$i^2 = j^2 = k^2 = ijk = -1. \tag{8}$$

We define two copies:

$$Q_8^{(L)} = \{\pm 1_L, \pm i_L, \pm j_L, \pm k_L\} \subset SU(2)_L, \tag{9}$$

$$Q_8^{(R)} = \{\pm 1_R, \pm i_R, \pm j_R, \pm k_R\} \subset SU(2)_R. \tag{10}$$

Then

$$Q_8^{(L)} \times Q_8^{(R)} \ \subset \ SU(2)_L \times SU(2)_R \ \subset \ SO(4) \ \subset \ SL(4,\mathbb{R}). \tag{11}$$

This 16-element subgroup is the natural finite "core" of the $SU(2)_L \times SU(2)_R$ electroweak stage in the $SL(4,R)$ real-form chain.

3 Triplet extensions: binary tetrahedral vs broken triplet

We now adjoin triplet symmetries separately to $Q_8^{(L)}$ and $Q_8^{(R)}$.

3.1 Unbroken triplet symmetry in $SU(2)_R$

The binary tetrahedral group $2T \subset SU(2)$ is a 24-element extension of Q_8 obtained by adjoining

$$t = \frac{-1 + i + j + k}{2}, \tag{12}$$

and its conjugates. We take

$$2T_R = \langle Q_8^{(R)}, t_R \rangle \subset SU(2)_R, \tag{13}$$

with t_R the right-handed analogue of t. This gives an unbroken triplet symmetry on the right-handed side.

3.2 Broken triplet symmetry in $SU(2)_L$

On the left-handed side we pick i_L as special and adjoin

$$s_L = \frac{-1 + i_L\sqrt{3}}{2}, \tag{14}$$

which generates a different 24-element extension

$$G_L = \langle Q_8^{(L)}, s_L \rangle \subset SU(2)_L, \tag{15}$$

with a distinguished i_L direction corresponding to the third component of weak isospin.

3.3 Product group and order 288 subgroup

The direct product

$$H = G_L \times 2T_R \subset SU(2)_L \times SU(2)_R \tag{16}$$

has order

$$|H| = 24 \times 24 = 576. \tag{17}$$

Quotienting by a central $\mathbb{Z}_2$ (identifying $(-1_L, -1_R)$ with $(1_L, 1_R)$) gives a subgroup of order

$$|H/\mathbb{Z}_2| = 288, \tag{18}$$

which matches the finite group structure described by Wilson: a normal subgroup of order 16 (the $Q_8^{(L)} \times Q_8^{(R)}$ core) and a quotient of order 18 ($S_3 \times \mathbb{Z}_3$ arising from the triplet symmetries).

Thus we obtain an explicit finite subgroup

$$H_{288} \subset SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4, \mathbb{R}), \tag{19}$$

realising the finite quantisation of the $SU(2)_L \times SU(2)_R$ stage in the $SL(4, R)$ real-form chain.

4 Embedding into the $SL(4, R)$ real-form chain

In the twistor unification framework we have the real-form chain

$$SL(4, R) = \mathrm{Spin}(3, 3) \supset SO(3, 3) \supset SO(4) \simeq \frac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}, \tag{20}$$

with the electroweak decomposition

$$SO(3,3) \;>\; SO(4) = SU(2)_L \times SU(2)_R \;\to\; SU(2)_L \times U(1)_{EM}. \tag{21}$$

The finite subgroup H_{288} sits precisely at the $SU(2)_L \times SU(2)_R$ level, providing a discrete quantisation of the left/right electroweak symmetry within the $SL(4,R)$ branch.

5 Colour vs flavour and $SL(3,R) \to SU(3)_{\text{eff}}$

In the strong sector, neutrino–antineutrino composite shells naturally lead to the split real form

$$SL(3,R), \tag{22}$$

with Lie algebra

$$\mathfrak{sl}(3,R) = \{A \in M_3(\mathbb{R}) \mid \operatorname{tr}(A) = 0\}. \tag{23}$$

Compactification of this split form yields an effective colour group

$$SL(3,R) \;\to\; SU(3)_{\text{eff}}, \tag{24}$$

with gluons represented as composite excitations of the neutrino–antineutrino vacuum shells.

In this picture, non-commuting triplet symmetries in the finite group H_{288} are naturally modelled, at the continuous level, by $SL(3,R)$ rather than by a commuting torus in $U(3)$. The emergent $SU(3)_{\text{eff}}$ is a compactification of $SL(3,R)$, not a fundamental gauge group, consistent with the view that colour is not an observable symmetry but an effective description of strong interactions.

6 Relation to $SL(4,C)$ and twistor real forms

The global complex symmetry

$$SL(4,C) = \text{Spin}(6,C) \tag{25}$$

admits three real forms relevant to four-dimensional physics:

$$SU(4), \qquad SL(4,R), \qquad SU(2,2), \tag{26}$$

with the twistor chain

$$SU(2,2) \subset SL(4,C) \supset SL(4,R), SU(4). \tag{27}$$

The finite subgroup $H_{288} \subset SL(4,R)$ can be viewed as a discrete twistor-like structure for the $SL(4,R)$ branch, complementing the continuous twistor geometry of the $SU(2,2)$ real form and the compact internal symmetry of the $SU(4)$ real form. The following table (on the last page of this paper) summarizes Dr Wilson's own analysis of the physics of his quantization method.

References

Wilson, R. A, Quantisation of SL(4,R), August 2, 2026. https://robwilson1.wordpress.com/2
of-sl4r/

Cook, B. B., Unification by SL(4,C) Twistors, July 31, 2026 https://ai.vixra.org/pdf/2608.00

Group	Description	Physical interpretation
Q_8	Quaternion group (order 8). Required in quantum mechanics: non-commutativity and elements squaring to -1. Represents the minimal discrete spin structure.	Finite skeleton of $SU(2)_L$ and $SU(2)_R$. Forms the 16-element core $Q_8^{(L)} \times Q_8^{(R)} \subset SO(4) \subset SL(4,R)$, encoding chiral spin and parity structure.
Z_3	Cyclic group of order 3. Minimal mathematical structure for three flavours of electrons (e, μ, τ) and, more generally, three fermion generations.	Discrete flavour triplet underlying the strong-sector chain $SL(3,R) \to SU(3)_{\text{eff}}$, where colour and flavour are emergent rather than fundamental.
$Z_3 \rtimes Q_8$	Semidirect product where Z_3 acts on Q_8, breaking the symmetry of Q_8 and coupling the flavour triplet to the quaternionic spin structure.	Left-handed weak interaction symmetry. Realised as the broken triplet $G_L \subset SU(2)_L$. Produces weak-isospin asymmetry and parity violation, with a distinguished left-handed direction corresponding to T_3.
$Q_8 \rtimes Z_3$	Semidirect product where Q_8 acts on Z_3, breaking the symmetry of Z_3 and putting a direction on the flavour triplet.	Right-handed flavour symmetry. Realised as the unbroken triplet $2T_R \subset SU(2)_R$. Gives three right-handed flavour directions and a mass hierarchy, by orienting the triplet of generations.
S_3	Permutation group of three objects, acting on the flavour triplet by permuting the three generations.	Discrete analogue of CKM-like mixing: flavour permutations appear as part of the quotient $H_{288}/(Q_8^{(L)} \times Q_8^{(R)})$, encoding generation mixing in the finite-group framework.
$\mathbb{Z}_3$ (phase)	Cyclic group of order 3 arising as a phase symmetry on the flavour triangle (clockwise vs anticlockwise rotations).	Provides flavour directionality and contributes to mass hierarchy. The same $\mathbb{Z}_3$ structure appears in the compactification $SL(3,R) \to SU(3)_{\text{eff}}$, linking finite flavour phases to emergent colour.
H_{288}	Combined structure $(G_L \times 2T_R)/\mathbb{Z}_2$, a finite subgroup of $SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4,R)$, with normal core $Q_8^{(L)} \times Q_8^{(R)}$ and quotient $S_3 \times \mathbb{Z}_3$.	Full discrete electroweak + flavour + proto-colour symmetry in the $SL(4,R)$ branch. Provides a discrete twistor-like structure: all allowed discrete chirality flips, weak-isospin asymmetries, flavour permutations, mass-hierarchy directions, and proto-colour phases acting on an $SL(4,R)$ twistor-spinor.